GOLDEN OLDIE 18

On the three-dimensional spaces which admit a continuous group of motions

Luigi Bianchi

Memorie di Matematica e di Fisica della Societa Italiana delle Scienze, Serie Terza, Tomo **XI**, pp. 267–352 (1898).

EDITOR'S NOTE

This article methodically studies (locally) the symmetry and isometry classes of all 3-dimensional Riemannian manifolds. For each of the possible orbit dimensions 1 and 2 (intransitive actions) and 3 (transitive actions) and for each possible symmetry class of group actions, explicit canonical coordinate expressions are derived for the full Killing vector Lie algebra and the metric by solving the Killing equations. A representative line element is then given parametrizing the isometry classes of a given symmetry type modulo constant conformal transformations, and specializations which admit higher symmetry are studied. For the case of simply transitive 3-dimensional isometry groups, this classification of metrics by symmetry class coincides with the classification into nine isomorphism classes of the isometry groups themselves (Bianchi types I – IX), now known together as the Bianchi classification.

This article followed soon after Lie's classification over the complex numbers of all Lie algebras up to dimension 6 and Killing's discovery of his famous Killing equations at the end of the nineteenth century. All of Bianchi's work was well known by the mathematician Luther P. Eisenhart (1876–1965), a professor, chair, dean and important educator in the Princeton University Mathematics Department from 1900 to 1945 [1], who served as a principal source of English language discussion of much of the early work in Riemannian geometry and Lie group theory through his two books *Riemannian Geometry* (1925) [2] and *Continuous Groups of Transformations* (1933) [3], which contain numerous references to Bianchi's work. As a differential geometer, Eisenhart occasionally helped Einstein and certainly contributed to the enthusiasm for relativity at Princeton.

The results of this Bianchi article were extended and brought to the attention of the relativity community in 1951 by Abraham Taub just after their first use in two special applications by Gödel. Taub got his Ph.D. in Mathematics at Princeton University in 1935 under H.P. Robertson, during the time (1933–1939) in which the Institute for Advanced Study was founded but initially housed in the Princeton University mathematics building where Taub had learned his differential geometry from Eisenhart and worked with both Oswald Veblen and John von Neumann, two of the three mathematicians stolen from the university as the original members of the Institute with Einstein when it opened in 1933

[4]. Later Gödel, Einstein's best friend and a fellow member of the Institute, was led by philosophical questions about time [5, 6] to consider studying rotating universe models in the late 1940s, leading to the first application of Bianchi's homogeneous spaces in general relativity (types III, VIII [7]), shaking up the physics community with the strange new properties of his stationary rotating solution (1949) [8], followed by a summary of results he published in 1950 without proof about rotating and expanding cosmological solutions (type IX) [9]. Taub was Veblen's assistant in 1935–36 and a visiting member of the Institute in 1947–48 [10] and soon after announced his own work at the same conference at which results of Gödel's second investigation were presented, shortly later in 1951 publishing explicit formulas for the spatially homogeneous spacetime metrics corresponding to all of Bianchi's nine symmetry types and the vacuum Einstein equations for these cosmological models in a discussion (like Gödel) motivated by the desire to find solutions violating Mach's principle [11].

These Bianchi models, as they later came to be called, were revived in the late 1950's by Heckmann and Schücking (later summarized in their chapter [12] in *Gravitation, an Introduction to Current Research* edited by Louis Witten, father of Edward). István Ozsváth continued this work in the next decade [13, 14], during which time David Farnsworth and Roy Kerr (1966) introduced the modern Lie group description of homogeneous spaces in relativity [15], while C.G. Behr (1968) introduced the modern Lie algebra version of the Bianchi (Lie) classification of 3-dimensional Lie algebras using the irreducible parts of the structure constant tensor under linear transformations [16]. Meanwhile Ronald Kantowski (1966) explored for the first time the spatially homogeneous (Kantowski-Sachs) models with no simply transitive subgroup [17], a spatial geometry thoroughly studied by Bianchi (§9) but for some reason omitted in his final summary, while Ellis (1964, 1967) pioneered the application of modern tetrad methods to cosmological models in his study of locally rotationally symmetric dust, involving the whole class of Bianchi symmetry types admitting multiply transitive groups.

At the close of the 1960's an ongoing investigation into the nature of the initial singularity of the universe by the Russian school of Lifshitz and Khalatnikov, later joined by Belinsky, independently led to the general Bianchi models in describing how the spacetime metric behaves along timelike curves approaching a "generic" spacelike singularity in some limiting approximation that was then controversial. At about the same time a study of the chaotic behavior they discovered was begun by Misner, who used Hamiltonian methods to explore the Bianchi type I and Bianchi type IX (Mixmaster) universe dynamics in the USA. While Ellis and MacCallum [18] approached the Bianchi models from an orthonormal frame point of view in England, Misner's Hamiltonian studies were continued by his student Michael P. Ryan for the general Bianchi model case, later summarized in 1975 in the only book devoted specifically to Bianchi cosmology [19], in whose bibliography references to the above-mentioned but uncited work may be found by year of publication. Bogoyavlensky and Novikov pioneered the application of the qualitative theory of differential equations to the dynamics of general Bianchi models (1973); more recent work in this direction is described in the book *Dynamical Systems in Cosmology* [20], where references to their work may be found.

The Bianchi models are spatially homogeneous spacetimes, the spatial sections of which are homogeneous Riemannian 3-manifolds of a fixed Bianchi type, and usually they are interpreted as cosmological models. While generally spatially anisotropic, they contain the spatially homogeneous and isotropic Friedmann-Robertson-Walker models as special cases for certain symmetry types, and enable Einstein's equations or similar gravitational field equations to be reduced from partial to ordinary differential equations, which are much easier to study. Besides providing more generalized models of certain aspects of the early universe, they have also been invaluable in helping to understand features of general relativity itself by providing an arena where certain questions can be more easily investigated. The most recent and sophistocated new work on spatially homogeneous cosmologies and their spatial geometries involves the Teichmüller space analysis of the dynamical degrees of freedom and Hamiltonian structure for spatially compactifiable models[21].

In 1972 (during the "golden age of relativity" at Princeton) when John Wheeler was bringing in proofs of his new book *Gravitation* with coauthors Charles Misner and Kip Thorne [22] to my sophomore Modern Physics class at Princeton University, junior Jim Isenberg was recruiting students to fill the quota for a student initiated seminar on Differential Geometry for General Relativity to be offered by Wheeler's collaborator Remo Ruffini. Following Wheeler's teaching style, Ruffini wanted to get the students more involved by doing special projects, and one suggestion was for a student to help him translate into English the original papers of Bianchi on homogeneous 3-spaces. Somehow I volunteered, but it immediately became clear that this was very inefficient so I boned up on some elementary Italian based on 3 prior semesters of college Spanish and tackled the job during the summer while working during the day as a carpenter with my dad.

This was followed by a junior paper on Bianchi cosmology and later a senior thesis begun in 1973 when Ruffini (my advisor) was excited by his investigation of the orbits of particles in rotating black hole spacetimes with another undergraduate from his seminar (Mark Johnston, whose graphics led to the famous Marcel Grossmann Meeting logo). Curious about rotating cosmologies, Ruffini wondered about talking to Gödel himself about the problem. Looking him up in the phone book (still naive times for celebrities), Ruffini found him, called him up and arranged for me to meet him at his office at the Institute, where he informed me about recent work by Michael Ryan that I had not been aware of, initiating my own work in the dynamics of Bianchi cosmology. Gödel, though his only published work in relativity was over 20 years old at the time, had still been following current developments related to it.

Ruffini later channeled me toward grad school at the University of California at Berkeley to work with Abe Taub just before his retirement in 1978. However, the Bianchi translation, although it had been typed up by a Jadwin Hall secretary in 1973, never found a wider audience, and sat for 25 years until Andrzej Krasiński asked me if I might translate the long article for this series, not knowing that it had essentially already been done (but which needed conversion to a compuscript and a polishing of my translation with my Italian improved by 20 years of regular visits to Rome). Unfortunately by this time (1999), Taub was in failing health and then passed away and could no longer be consulted to unravel more of the interesting history of the personalities at Princeton tied together by Bianchi's work.

However, this question led to my volunteer project to put *The Princeton Mathematics Community in the 1930s: An Oral History Project* [23] on the Princeton University Library web together with background materials that should be of interest to those curious about the community that welcomed Einstein with the founding of the Institute for Advanced Study in Princeton.

Finally this project could not have been completed (2001) without LaTeX, which allowed me to typeset a long formula-dense document, nor without the encouragement and invaluable editorial assistance of Andrzej Krasiński, who went beyond the call of duty in many rounds of proofreading comparing my document to the original Italian manuscript to ensure as accurate a reproduction as possible in every detail.

Commentary on Bianchi's Article in Translation

Terminology Bianchi uses the term "group" to mean "transformation group" or a group action on a manifold, expressed in terms of local coordinates on the group manifold and the manifold on which it acts, and he specifies such groups G_r (dimension r) by giving a basis of the Lie algebra of generating vector fields (called "infinitesimal transformations"), using the notation $G_r \equiv (X_1 f, \ldots, X_r f)$, where $Xf = \sum_i^{1..n} \xi_i \partial f / \partial x_i$ denotes the action of a vector field on an arbitrary function f by differentiation and his square bracket delimiters have been replaced by parentheses to lessen confusion with the modern commutator notation. Coordinate (and index) labels are subscripted in his notation: x_i. The modern Christoffel symbols of the first and second kind $[ij,k]$ and $\left\{ {k \atop ij} \right\}$ have replaced the original symbols $\left[{ij \atop k} \right]$ and $\left\{ {ij \atop k} \right\}$ in use at the time (apparently introduced by Eisenhart to conform with the Einstein index convention [2]). Bianchi's commutator (Lie bracket) notation $(X_1 X_2) = X_1 f$ which uses parentheses but no comma, with no arbitrary function to the right of the commutator (although both comma and function appear in his later work [24]), has been modernized to the square bracket convention $[X_1, X_2]f = X_1 f$. Two equivalent group actions (in the coordinate representation: related by invertible joint coordinate transformations on the group manifold and manifold on which it acts), are called "similar" by Bianchi, while two metrics are called "similar" if they are locally isometric modulo a constant conformal factor. (Bianchi uses the term "applicable", which has been updated to "isometric.") The transformations of a group acting as isometries of a metric are called "motions." The version of this article published in his collected works has more complete footnotes (consecutively numbered, first name initials added) which have been used here, and a correction in proof (rewording of the beginning of the next to last sentence of §21) was incorporated into the text as done there, together with a correction to the sign of equation (62b) which allowed the deletion of several lines at the end of §19, and a few other minor corrections. Multiple equations grouped together by an expanded left brace delimiter in the original have been distinguished here by a letter following the equation number or the brace has simply been omitted when unnecessary, and some displayed equations have been incorporated into the text. (As a consequence of this modification of equation numbers, the references to those numbers in the text were also modified but not marked by footnotes.) Finally a number of index typo's from the original articles have been corrected in translation.

The group generated by the commutators of the generating vector fields is called the derived group. The numerical scheme characterizing the Bianchi classification of simply transitive 3-dimensional group actions is a simple one based on the sequence of successive derived groups starting from the original one, as described in §198 (I sette tipi di composizioni dei G_3 integrabili, The seven types of compositions of integrable G_3's) and §199 (I due tipi dei G_3 nonintegrabili, The two types of nonintegrable G_3's) of his book on continuous groups [24]. This is Bianchi's version of Lie's classification of equivalence classes of 3-dimensional Lie algebras over the complex numbers, refined for equivalence over the real numbers. (In a similar way Lie essentially classified all Lie algebras up to dimension 6, with the 4-dimensional case done explicitly in all detail.) Luther Eisenhart's book [3] is an excellent source of information for the terminology and Lie group details of this early generation of geometers.

Preliminaries The article begins with the Killing equations for an n-dimensional Riemannian manifold (§1) and briefly treats the one Killing vector case (§2). (Bianchi uses the notation X for $\pounds_X$.) Then the classification of 2-dimensional Riemannian manifolds with simply transitive isometry groups (G_2) is reviewed (§3), with only two discrete transformation group types: Abelian and non-Abelian, leading to the negative and zero curvature cases, both of which have 3-dimensional complete isometry groups. Since the derived group of a G_2 generated by the commutators of the generating vector fields must be 0 or 1-dimensional in two dimensions, choosing X_1 to span its Lie algebra gives the canonical form of the non-Abelian case commutation relations: $[X_1, X_2] = \epsilon X_1$, with $\epsilon = 1$. The Killing vectors $(X_1, X_2) = (e^{-\epsilon x_2}\partial/\partial x_1, \partial/\partial x_2)$ and the general forms of the metric and the invariant 1-forms are derived but not explicitly stated:

$$ds^2_{(2)} = \alpha\,(dx_1 + \epsilon x_1 dx_2)^2 + 2\beta\,(dx_1 + \epsilon x_1 dx_2)\,dx_2 + \gamma\,dx_2^2\ ,$$

with $\epsilon = 1$ describing the non-Abelian case and $\epsilon = 0$ the Abelian case.

This is then used in the case of 3-dimensional Riemannian manifolds with 2-dimensional intransitive isometry groups ($G_2 \equiv (X_1 f, X_2 f)$) acting simply transitively on 2-dimensional orbits (§4). The orbits are a family of geodetically parallel surfaces taken as x_1 coordinate surfaces in an adapted Gaussian normal coordinate system with orthogonal geodesics along the coordinate lines of x_1, while x_2, x_3 are adapted to the generators as above, leading to the general form

$$ds^2 = dx_1^2 + \alpha\,(dx_2 - \epsilon x_2 dx_3)^2 + 2\beta\,(dx_2 - \epsilon x_2 dx_3)dx_3 + \gamma\,dx_3^2\ ,$$

where the three independent components α, β, γ of the 2-metric in the invariant form basis are functions only of x_1.

Then the complete isometry groups possible for such intransitive actions (2-dimensional orbits) are described (§§5–11), which can only be at most 3-dimensional, forcing the surfaces to have constant curvature. This is the case of intransitive groups acting multiply transitively on 2-dimensional orbits. However, the additional isometries can lead to a transitive action, which is the case for the 4-dimensional isometry groups of the positive (§9) and negative (§11) curvature Kantowski-Sachs geometry, or the 6-dimensional isometry groups of constant positive (§8), zero (§8), or negative (§§8,10) curvature spaces.

Homogeneous 3-Manifolds These preliminary considerations are then used in the case of 3-dimensional simply transitive isometry groups. Such simply transitive actions are first introduced for any dimension, with a discussion of the coordinate representation of the Killing equations and their integrability conditions, the latter being satisfied identically by virtue of the generating Lie algebra Lie bracket relations (§12). Then Lie's classification of equivalence classes of 3-dimensional Lie algebras over the complex numbers is refined to the real case by adding several types, giving Bianchi's canonical form for the generating Lie algebra commutation relations for each type designated by consecutive Roman numerals I through IX, now known as the Bianchi classification (§13).

Types I through VIII all have a 2-dimensional subgroup $G_2 \equiv (X_1 f, X_2 f)$ acting simply transitively on 2-dimensional orbits, so the metric can be reduced to the standard form given above for intransitive actions on surfaces, with types I through VII belonging to the Abelian subgroup case and type VIII to the non-Abelian subgroup case.

Equations (E) of §14 are Killing's equations for this metric in the Abelian subgroup case, then applied in (F) to the third generating vector field X_3 whose Lie brackets with X_1, X_2 are given by equations (41). Specializing this pair of sets of equations to each symmetry type then leads to the complete symmetry group, including the coordinate representation of X_3 and additional independent Killing vector fields, and to explicit values for the three metric coefficient functions of x_1, from which one may easily read off the invariant 1-forms in terms of which the metric is expressed, though not done explicitly by Bianchi. The coordinate expressions for the metric and Killing vector fields are then specialized (by rescaling the surface coordinates x_2, x_3, by affine transformations of the surface parametrizing coordinate x_3, and by constant conformal transformations) to a simple canonical form with the minimum number of free parameters, which then parametrize the conformal equivalence classes of homogeneous 3-geometries (locally). The symmetry type I case of flat 3-space in orthonormal Cartesian coordinates with its translation symmetries is trivially obtained from these equations, with a 6-dimensional complete isometry group.

For the symmetry type II (§16), all metrics are conformally equivalent and have a 4-dimensional complete isometry group whose finite equations are given, corresponding to right multiplication of a unit upper triangular 3×3 matrix $X(x_1, x_2, x_3) = I_3 + x_3 e^2{}_1 - x_2 e^3{}_1 + x_1 e^3{}_2$ by $A(-a_1, -a_2, -a_3)^{-1}$.

For the symmetry type III (§17), a 1-parameter family of conformal equivalence classes is found, with the parameter n measuring the nonorthogonality of the surface coordinates x_2, x_3, and a 4-parameter complete group of isometries whose derived group $(X_1 f, X_3 f, X_4 f)$ is of type VIII, which acts transitively when $n \neq 0$ and intransitively when $n = 0$ (therefore appearing in the discussion of intransitive actions). The additional linearly independent Killing vector field $X_4 f$ depends on n. The proof that n parametrizes the conformal 3-geometry involves showing that two metrics with the same canonical form in two coordinate systems but with different values n and m of the parameter cannot be related by a coordinate transformation. The two 4-dimensional isometry groups must be equivalent by a theorem of Lie, but the accompanying canonical generating vector fields need only be transformed into each other by the coordinate transformation modulo a Lie algebra automorphism. The 4-parameter group of Lie algebra automorphisms is easily found, and with some more work, a 3-parameter group of coordinate transformations which transform the generators into each other, giving the equivalence transformation between the two isometry groups (§18). The partial derivatives of one set of coordinates with respect to the other can be read off from the transformation of the generating vector fields, and used to evaluate the transformation of the one metric into the other. Requiring the two metrics to be related by the same coordinate transformation modulo a constant conformal factor then forces the two parameters n and m to be the same modulo an irrelevant sign (§19). As noted above, the form (49) of the type III metric, changed in signature, slightly rescaled and with a special value of n, was used by Gödel for the timelike homogeneous sections of his stationary spacetime homogeneous solution.

For the symmetry type IV (§20), similarly a 1-parameter family of conformal equivalence classes is found with no additional Killing vector fields. The 4-parameter group of Lie algebra automorphisms is easily found, and then a 5-parameter family of coordinate transformations which induce them, and the essential nature of the parameter is again shown by transforming the metric (§21).

The symmetry type V immediately leads to an orthogonal coordinate representation of the constant negative curvature geometry with a 6-dimensional complete isometry group (§22) whose generators are derived as an example in §38.

The symmetry type VI (§§23,24) is entirely analogous to the type IV case. Bianchi does not distinguish the modern class A and class B types VI_0 and $\mathrm{VI}_{h\neq 0,-1}$, where the subscript h is the Behr parameter described below, differing from Bianchi's parameter $h \neq 0, 1$. Bianchi's $h = -1$, $h = 0$ and $h = 1$ limits of type VI give types VI_0, V and III respectively.

The symmetry type VII is split into types VII_1 and VII_2, corresponding to VII_0 and $\mathrm{VII}_{h\neq 0}$ in the modern Behr notation but with a different parameter h. The metric coefficients and X_3 are found (§§25,26), again with a 1-parameter family of conformal equivalence classes and with no additional Killing vector fields (except for the special case of type VII_0 corresponding to flat 3-space), and then a 4-parameter Lie algebra automorphism group is found and used to show the essential nature of the parameter (§27).

The symmetry type VIII then switches to the non-Abelian $G_2 \equiv (X_2 f, X_3 f)$ subgroup case (§28), where two cases arise. The simpler case with an additional Killing vector field is equivalent to the type III case $n \neq 0$, but the more general case in which no additional Killing vectors exist, the Gaussian normal coordinates lead to elliptic functions in the integration of the Killing equations for the metric coefficients and the third generator X_1 (§§29,30), where Bianchi's original notation for the elliptical functions lacked parentheses around their arguments. By choosing a more general coordinate x_1 whose coordinate lines are no longer orthogonal to the 2-surface orbits of the G_2 and which does not measure arclength along them, but for which X_1 has a relatively simple form ($A = 1$, $B = C = 0$ in equation (95) for X_1), expressions are found for the metric coefficients which are at most quadratic in the two nontrivial coordinates separately (§31).

Finally the symmetry type IX requires a similarly different approach. Canonical generating vector fields long known from Euler angle parametrizations of the rotation group are chosen and the Killing equations integrated to yield a 6-parameter family of metric coefficients from which one could read off the invariant 1-forms (§32). No discussion of conformal equivalence classes is given. Relying again on Lie, the case in which an additional independent Killing vector field X_4 exists is treated, leading to a 1-parameter conformal equivalence class of metrics which includes the special case of constant positive curvature (and a 6-dimensional complete isometry group) for a particular value of the parameter (§33), whose essential nature is shown in §35 after showing that no additional Killing vectors exist other than X_4 (§34).

That no other possibilities have been overlooked is shown in §36, relying on the the fact that no 5-dimensional isometry groups can exist as shown in §37. The final section summarizes the canonical form of the metric, Killing vector fields, and their Lie brackets for most of these possibilities, although the 4-dimensional isometry group with no simply transitive 3-dimensional subgroup case of §11 is curiously omitted, perhaps leading to the nearly two decade delay in its application to spatially homogeneous cosmological models, first done by Kantowski and Sachs [17].

Obtaining the Same Results Painlessly from a Modern Perspective At least in the general relativity literature, Farnsworth and Kerr [15] first published the modern description of a simply transitive symmetry group action as the natural left or right action of any Lie group on itself, moving the jargon away from the old fashioned simply transitive transformation group accompanied by an isomorphic reciprocal group to left and right translation on the group manifold. Choosing a left action for the symmetry action, the left invariant vector fields (Lie algebra of the Lie group) are the homogeneous ("invariant") vector fields, the left invariant (positive-definite) metric tensors on the group manifold are the homogeneous Riemannian metrics, with the corresponding Killing Lie algebra for this "homogeneity" action equal to the Lie algebra of right invariant vector fields. For spatially homogeneous spacetimes ("Bianchi cosmologies"), the induced metrics of the spatial hypersurfaces of homogeneity are isometric to left invariant Riemannian metrics on a fixed 3-dimensional spacetime isometry group.

Behr [16] was the first to then publish a simpler scheme for classifying the equivalence classes of 3-dimensional Lie algebras using the irreducible parts of the structure constant tensor under linear transformations rather than the more complicated derived group approach of Lie and Bianchi, exploiting the special properties of the duality operation in 3 dimensions: taking the natural dual of the covariant antisymmetric indices of the structure constant tensor $C^a{}_{bc}$ leads to a 2-covariant tensor density on the Lie algebra $C^{ab} = \frac{1}{2}C^a{}_{cd}\epsilon^{bcd}$ which can be decomposed into its symmetric $n^{ab} = C^{(ab)}$ and antisymmetric parts $C^{[ab]} = \epsilon^{abc}a_c$, and the dual of its antisymmetric part leads to a covector $a_c = \frac{1}{2}\epsilon_{cab}C^{ab}$ which the Jacobi identity shows must have zero contraction with the symmetric 2-tensor $n^{ab}a_b = 0$. When nonzero, this covector's self tensor product must then be proportional to the matrix of cofactors of the 2-covariant symmetric tensor with a scalar constant of proportionality $a_a a_b = \frac{1}{2}h\epsilon_{acd}\epsilon_{bfg}n^{cf}n^{dg}$. This notation was introduced by Ellis and MacCallum [18], who also coined the terms class A for the case $a_b = 0$ and class B for the case $a_b \neq 0$ (corresponding to unimodular and nonunimodular Lie algebras [25]). Diagonalizing the symmetric tensor density n^{ab} aligns a_b with one of the basis vectors in general, leading to a standard "diagonal form" for the 4 nonzero components of the structure constant tensor (of which at most 3 can be simultaneously nonzero), from which the equivalence class representative structure constants are obtained by quotienting out by the scale transformations of the Lie algebra basis vectors.

Jantzen [26] realized that explicit expressions for all the invariant vector fields and 1-forms, and hence for the homogeneity Killing vector fields and the general form of the metric, could be easily obtained from the generic expressions for the linear adjoint matrix group associated with a diagonal form basis of the Lie algebra, which generically has the same Lie algebra structure as the original Lie algebra in 3 dimensions, with limiting cases following by analytic continuation of the formulas valid in the general case. Similarly by considering the orbits of the easily constructed automorphism matrix groups on the space of inner products on the Lie algebra, one can algebraically determine the isometry classes of individual Bianchi symmetry types. This almost eliminates the need for solving any partial differential equations, the element responsible for the length of Bianchi's classification paper.

References

[1] Charles C. Gillispie, editor, *Dictionary of Scientific Biography*, Scribner's Sons, New York, 1970.

[2] Luther P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1925.

[3] Luther P. Eisenhart, *Continuous Groups of Transformations*, Princeton University Press, 1933.

[4] Ed Regis, *Who Got Einstein's Office?*, Addison-Wesley, New York, 1987.

[5] John W. Dawson, Jr, *Logical Dilemmas: The Life and Work of Kurt Gödel*, A.K. Peters, Wellesley, Massachussetts, 1997, Chapter 9.

[6] *Kurt Gödel: Collected Works*, Vols. I (1986), II (1990), edited by Solomon Feferman, Vol. 3 (1995), edited by Solomon Feferman, John W. Dawson, Jr., Warren Goldfard, Charles Parsons and Robert M. Solovay, The Clarendon Press, Oxford University Press, New York; see especially the unpublished manuscript of his famous Institute for Advanced Study lecture of May 7, 1949 in Vol. 3 and his preceding discussion of the relationship between relativity and Kantian philosophy which motivated his cosmology work.

[7] István Ozsváth, "Dust-Filled Universes of Class II and Class III," *J. Math. Phys.* **11**, 2871 (1970); Robert T. Jantzen, "Generalized Quaternions and Spacetime Symmetries," *J. Math. Phys.* **23**, 1741 (1982); Gödel's Lorentzian 3-metric obtained setting $dx_3 = 0$ (a trivial translation symmetry coordinate) coincides with Bianchi's equation (49) with $(n, dx_3, dx_2, dx_1^2) \to (\sqrt{2}, dx_0, dx_2/\sqrt{2}, -dx_1^2)$, but this locally rotationally symmetric type III geometry coincides with a locally rotationally symmetric type VIII geometry slightly deformed from the bi-invariant Cartan-Killing metric on the

group, which Gödel exploits using a quaternion-like representation, later discussed by Ozsváth.

[8] Kurt Gödel, "An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation," *Reviews of Modern Physics* **21**, 447 (1949); *Gen. Rel. Grav.* **32**, 1409-1417 (2000).

[9] Kurt Gödel, "Rotating Universes in General Relativity Theory," *Proceedings of the International Congress of Mathematicians*, Cambridge, Mass. 1950, Vol. 1, 175, Amer. Math. Soc., R.I., 1952; *Gen. Rel. Grav.* **32**, 1419-1427 (2000).

[10] *Essays in General Relativity: a Festschrift for Abraham Taub*, edited by Frank J. Tipler, Academic Press, New York, 1980.

[11] Abraham Taub, "Empty Spacetimes Admitting a Three-Parameter Group of Motions," *Proceedings of the International Congress of Mathematicians (Cambridge, Mass., 1950), p.655*; *Annals of Mathematics* **53**, 472 (1951); to appear in *Gen. Rel. Grav.* **33**, (2001).

[12] O. Heckmann and E. Schücking, "Relativistic Cosmology," in *Gravitation, an Introduction to Current Research*, edited by Louis Witten, Wiley, New York, 1962.

[13] I. Ozsváth and E. Schucking, "Finite Rotating Universes," *Nature* **193**, 1168 (1962).

[14] I. Ozsváth, "Spatially Homogeneous World Models," *J. Math. Phys.* **11**, 2860 (1970); "Spatially Homogeneous Rotating World Models," *J. Math. Phys.* **12**, 1078 (1971).

[15] D.L. Farnsworth and R.P. Kerr, "Homogeneous, Dust-Filled Cosmological Solutions," *J. Math. Phys.* **7**, 125 (1966).

[16] F.B. Estabrook, H.D. Wahlquist, and C.G. Behr, "Dyadic Analysis of Spatially Homogeneous World Models," *J. Math. Phys.* **9**, 497 (1968).

[17] Ronald Kantowski, "Some Relativistic Cosmological Models" (Ph.D. thesis 1966), *Gen. Rel. Grav.* **30**, 1663 (1998); R. Kantowski and R.K. Sachs, "Some Spatially Homogeneous Anisotropic Relativistic Cosmological Models," *J. Math. Phys.* **7**, 443 (1966).

[18] G.F.R. Ellis and M.A.H. MacCallum, "A Class of Homogeneous Cosmological Models," *Commun. Math. Phys.* **12**, 108 (1969).

[19] Michael P. Ryan, Jr. and Lawrence Shepley, *Homogeneous Relativistic Cosmologies*, Princeton University Press, Princeton, 1975.

[20] *Dynamical Systems in Cosmology*, edited by J. Wainwright and G.F.R. Ellis, Cambridge University Press, Cambridge, 1997.

[21] T. Koike, M. Tanimoto and A. Hosoya, "Compact Homogeneous Universes," *J. Math. Phys.* **35**, 4855 (1994); M. Tanimoto, T. Koike and A. Hosoya, "Dynamics of Compact Homogeneous Universes," *J. Math. Phys.* **38**, 350 (1997); "Hamiltonian Structures for Compact Homogeneous Universes," *J. Math. Phys.* **38**, 6560 (1997).

[22] Charles W. Misner, Kip S. Thorne and John A. Wheeler, *Gravitation*, Freeman, San Francisco, 1973.

[23] *The Princeton Mathematics Community in the 1930s: An Oral History Project*, Charles C. Gillispie, Albert W. Tucker, William Aspray, and Frederik Nebeker, Princeton University, 1985; now on-line at `http://www.princeton.edu/mudd/math`.

[24] Luigi Bianchi, *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni* (Lectures on the theory of finite continuous transformation groups (1902-1903)), Spoerri, Pisa, 1918.

[25] John Milnor, "Curvatures of Left Invariant Metrics on Lie Groups," *Adv. Math.* **21**, 293 (1976).

[26] R.T. Jantzen, "The Dynamical Degrees of Freedom in Spatially Homogeneous Cosmology," *Commun. Math. Phys.* **64**, 211 (1978).

Acknowledgement The Editors of GRG are grateful to F. de Felice for his help in obtaining the publisher's permission for the translation and reprinting.

—Robert Jantzen

Department of Mathematical Sciences, Villanova University, Villanova, PA 19085 USA

Brief biography

Born in Parma, Italy on January 18, 1856, Luigi Bianchi began his mathematics studies at the Scuola Normale Superiore of Pisa in 1873 and then became a student of Ulisse Dini and Enrico Batti at the University of Pisa where he got his mathematics degree with distinction in 1877, with a dissertation on applicable (isometric) surfaces. After postgraduate studies in Pisa, Munich and then Göttingen where he studied with Felix Klein, he returned to Pisa to become a professor at the Scuola Normale in 1881 and was appointed as the chair in projective geometry in 1896. In the same year he became chair of analytic geometry at the University of Pisa and was later appointed as the director of the Scuola Normale Superiore of Pisa in 1918, holding both positions until his death in Pisa in 1928 [1, 2]. He had also been an editor of *Annali di Matematica pura ed applicata* and a member of the Accademia Nazionale dei Lincei.

His mathematical contributions, published in eleven volumes by the Italian Mathematical Union [3], cover a rather wide range of topics. In the field of Riemannian geometry, he is most well known for his discovery of the "Bianchi identities" satisfied by the Riemann curvature tensor [4] (1902). In 1898 Bianchi published his complete classification of the isometry classes of Riemannian 3-manifolds [5, 6], the more well known symmetry types categorized by his famous nine types identified by the Roman numerals I–IX, building upon the theory of continuous groups just developed by Sophus Lie [7, 8, 9] (1883–93) and the Killing equations found by W. Killing (1892) [10] a few years earlier.

Bianchi played an important role in the generations of mathematicians of the late 1800's and early 1900's who developed differential and Riemannian geometry and transformation group theory and their applications after their introduction by Gauss and Riemann and Lie, improved by the tensor analysis methods of Gregorio Ricci-Curbastro (developer of "Ricci Calculus" and also a student of Dini at the same time as Bianchi) and Tullio Levi-Civita (himself a former student of Ricci), which in turn influenced the development and birth (1915) of the new field of Einstein's general relativity. After Bianchi's death, his former student Guido Fubini characterized much of Bianchi's work as being a careful investigation of the many cases which can occur in answering a given mathematical question [11], certainly a fitting description of his long article categorizing 3-geometries with symmetry.

References

[1] Charles C. Gillispie, editor, *Dictionary of Scientific Biography*, Scribner's Sons, New York, 1970.

[2] MacTutor History of Mathematics Archive:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Bianchi.html

[3] Luigi Bianchi, *Opere* (The Collected Works of Luigi Bianchi), Rome, Edizione Cremonese, 1952.

[4] Luigi Bianchi, *Rend. Accad. Naz. dei Lincei* **11**, 3 (1902).

[5] *Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti* (On 3-Dimensional Spaces Which Admit a Continuous Group of Motions), *Memorie di Matematica e di Fisica della Societa Italiana delle Scienze*, Third Series, Vol. XI (1898), pp. 267–352; reprinted in: *Opere* (The Collected Works of Luigi Bianchi), Rome, Edizione Cremonese, 1952, vol. 9, pp. 17–109.

[6] Luigi Bianchi, *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni* (Lectures on the theory of finite continuous transformation groups (1902-1903)), Spoerri, Pisa, 1918.

[7] Sophus Lie and Georg Scheffers, *Vorlesungen über Continuierliche Gruppen mit Geometrischen und Anderen Anwendungen* (Lectures on Continuous Groups with Geometric and Other Applications), Leipzig, Germany, 1983 [reprinted by Chelsea Pub Co, Bronx, NY 1971].

[8] Sophus Lie (with Friedrich Engel), *Theorie der Transformationsgruppen* (Theory of Transformation Groups), Volumes 1–3, Liepzig, Germany, 1888,1890,1893 [reprinted by Chelsea Pub Co, Bronx, NY 1970].

[9] Lie Groups: History, Frontiers and Applications Volume 1: Sophus Lie's 1880 Transformation Group Paper, translated by Michael Ackermann, comments by Robert Hermann, Math Sci Press, Brookline, Massachussetts, 1976.

[10] W. Killing, "Über die Grundlagen der Geometrie" (On the Foundations of Geometry), *Journ. für die r. und ang. Math. (Crelle)*, **109**, pp. 121–186 (1892).

[11] Guido Fubini, "Luigi Bianchi e la sua opera scientifica," in *Annali di matematica pura ed applicata* **62**, pp. 45–81 (1929).

by Robert Jantzen