

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad). **You may use technology for row reductions, matrix inverses, determinants, plotting and root finding without showing intermediate steps. Print** the requested technology plots, labeling them and **annotating them appropriately by hand** and attach to the end of your test. All differential equations should be solved "by hand" unless otherwise specified.

1. The displacement $x(t)$ of a damped harmonic oscillator system satisfies

$$m x''(t) + c x'(t) + K x(t) = F(t).$$

a) Let $m = 9$, $c = 6$, $K = 10$, $F(t) = 0$ and consider the initial conditions $x(0) = 1$, $x'(0) = -1$. What are the natural frequency ω_0 , natural decay time τ_0 , the quality factor $Q = \omega_0 \tau_0$ and the natural period $T_0 = 2\pi/\omega_0$ for this system?

What are the solution frequency ω_1 and decay constant τ_1 ? (Give both exact and numeric values to 3 decimal places.)

) Evaluate the initial value problem solution by hand and use calculus to determine the largest displacement from equilibrium for $t > 0$ (maximize $|x(t)|$) and when it occurs, to 4 decimal place accuracy. Evaluate the envelope functions for this decaying oscillation and plot your solution together with the envelope functions for an appropriate decay window starting at $t = 0$, and mark the critical point corresponding to maximum displacement and label that point by its numerical coordinates [plot#1].

b) Let $m = 9$, $c = 6$, $K = 10$, $F(t) = 37 \sin(\omega t)$, $\omega \geq 0$ and evaluate the steady state solution by hand to explore resonance.

Evaluate and **plot** the (simplified!) amplitude $A(\omega)$ of the steady state solution for nonnegative frequency, showing it reaching its asymptotic value at large frequency [plot#2]. Use calculus to find the exact critical point at which the resonance peak occurs, and give the numerical values of its coordinates to 4 decimal places. Label this point in your plot. Evaluate also the ratio $A(\omega_{peak})/A(0)$ and compare with Q .

c) For $\omega = 1$, evaluate the phase-shifted form of the steady state solution (use a diagram!) and then evaluate the phase shift relative to the driving sine function phase, namely $\Delta\delta = \delta - \frac{\pi}{2}$. What fraction of a cycle does this relative shift represent?

2. $x_1'(t) = -9x_1(t) + x_2(t)$, $x_2'(t) = -5x_1(t) - 7x_2(t)$, $x_1(0) = 5$, $x_2(0) = 5$.

a) Write down the Maple solution of this initial value problem, simplified to integer coefficients.

b) Rewrite this system of DEs **and** its initial conditions explicitly in matrix form for the vector variable $\vec{x} = \langle x_1, x_2 \rangle$ as a column matrix (using the actual matrices, not their symbols), identifying the coefficient matrix A .

c) Derive by hand its eigenvalues λ_{\pm} and eigenvectors \vec{b}_{\pm} , $B = \langle \vec{b}_+, \vec{b}_- \rangle$, and check that they agree with Maple.

d) Find the general solution of the DE system in explicitly real form.

e) Find the solution of the initial value problem.

f) Plot x_1 and x_2 versus t for 8 characteristic times of the exponential factor starting at $t = 0$, labeling the two graphs [plot#3].

3. Consider the following open 3 tank problem:

$$x_1'(t) = -x_1(t), x_2'(t) = x_1(t) - \frac{1}{2}x_2(t), x_3'(t) = \frac{1}{2}x_2(t) - \frac{1}{3}x_3(t), x_1(0) = 6, x_2(0) = 0, x_3(0) = 0.$$

a) Write down the DE system and its initial condition in explicit matrix form, identifying the coefficient matrix A .

b) Evaluate the characteristic equation and determine the ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

- c) Solve the equations using the reduction algorithm necessary to find an eigenbasis matrix $B = \langle \vec{b}_1 | \vec{b}_2 | \vec{b}_3 \rangle$.
- d) Re-express the DE system in the new variables $\vec{y} = B^{-1} \vec{x}$ and solve the resulting decoupled equations.
- e) Impose the initial conditions on your general solution and express it in scalar form $x_1 = \dots$, etc
- f) Use calculus to find the exact value of the maximum of x_2 .
- g) Plot the three solution curves versus t for $t = 0 \dots T$ for an appropriate decay window which shows clearly their asymptotic behavior, labeling the three graphs [plot#4]. What is the longest characteristic time for the three modes here? How does it compare to your choice of window?

4. $x_1'(t) = -x_2(t)$, $x_2'(t) = x_1(t) - \frac{5}{2} x_2(t)$, $x_1(0) = -2$, $x_2(0) = 5$.

- a) Identify the coefficient matrix for this system of DEs and use Maple's Eigenvector result to write down a new basis for R^2 consisting of eigenvectors \vec{b}_1 , \vec{b}_2 of this matrix. Order the real eigenvalues $\lambda_1 \leq \lambda_2$ by increasing value.

Identify $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$.

- b) Evaluate the new coordinates $\langle y_1, y_2 \rangle$ of the point $\langle x_1, x_2 \rangle = \langle -2, 5 \rangle$ with respect to this basis of eigenvectors.
- c) Using the eigenbasis, solve this initial value problem by hand, showing all steps. Make sure the final result agrees with Maple.
- d) Use technology to plot a directionfield for this DE with the solution curve through the single initial data point [plot#5], and (by hand if necessary) include the lines through the two eigenvectors representing the two subspaces of eigenvectors. Choose an appropriate window that shows everything clearly without wasting additional window space, except to see the new coordinate axes extend slightly beyond the details of the plot. By hand label these lines by their new coordinate labels, draw in and label the eigenvectors and the initial data vector $\vec{x}(0)$ themselves as arrows, and include the parallelogram projection of the latter vector onto the new coordinate axes, i.e., draw the parallelogram parallel to the new coordinate axes with the initial data vector as its main diagonal.
- e) Which of the two characteristic times in this problem is longer and hence determines the timescale on which the asymptotic limit is "reached". What are the exact and numerical values of the two variables at 5 times this characteristic time?

Advice. When in doubt about how much work to show, show more. Explain using words if it helps. Think of this take-home test as an exercise in "writing intensive" technical expression. Try to impress bob as though it were material for a job interview. In a real world technical job, you need to be able to write coherent technical reports that other people can follow.

► solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:

"During this examination, all work has been my own. I have read the long instructions on the class web page. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature:

Date: