

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC). Maple may not substitute for any hand calculations unless explicitly stated, but use it to check each step if you want to be safe.

pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet in on top of your answer sheets as a cover page, with the first test page facing up:

"During this examination, all work has been my own. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: _____

Date: _____

1. Consider the power series $f(x) = \sum_{n=0}^{\infty} \left(\frac{(-1)^n \cdot (x-3)^n}{(n+1) 5^n} \right)$

- Determine the radius of convergence and the interval of convergence including endpoints. Justify your claims.
- Use Maple to evaluate $f(4)$ exactly and to 5 decimal places.
- Optional.** What does Maple evaluate this too, provided you assume x lies in the interval of convergence? Does its plot agree with your determination of the interval of convergence and its endpoints?

2. Use Taylor series to evaluate the limit $\lim_{x \rightarrow 0} \left(\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} \right)$. Explain your steps.

3. a) Evaluate the Taylor cubic polynomial $T_3(x)$ for the function $f(x) = \sqrt{x}$ at $x=4$ using the definition (hand evaluation!).

b) What is the numerical value of this approximating polynomial at $x=5$ to 5 decimal places.

c) Estimate the error in using this polynomial to approximate $\sqrt{5}$ (alternating series) and compare it with the numerical value of the exact error.

4. The formula $T_0 = 2\pi \sqrt{\frac{L}{g}}$ for the period of a simple pendulum we learn in high school and freshman physics only applies for small angles $\theta_0 \geq 0$ of maximum angular displacement from the downward vertical direction. A quick search of the web reveals that the exact formula has a correction factor

$$T = 2\pi \sqrt{\frac{L}{g}} \cdot f(x) \quad \text{with} \quad f(x) = \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n} \cdot (n!)^2} \right)^2 x^{2n},$$

where $x = \sin\left(\frac{\theta_0}{2}\right) \geq 0$. For a pretty large swing angle of 60° , note that $x = \frac{1}{2}$.

- Write out the first 3 nonzero terms of this series for $f(x)$, which form the Taylor fourth degree polynomial $T_4(x)$. Notice that the first term is 1 since this formula needs no correction at very small angles.
- Evaluate this numerically to 4 decimal places for an angle of 60° .
- Derive the radius of convergence R of this series representation of $f(x)$ in x using explicit limit notation. What angle θ_0 does this maximum value of x correspond to? Can you think of any physical reason connected to the motion of the pendulum why this angle might cause trouble?

- d) Maple can evaluate the sum of this infinite series for $f(x)$ exactly for any value of x for which the series converges. What is the numerical value of the series for a 60° degree angle?
- e) We only need to evaluate $f(x)$ correct to 2 decimal places to evaluate the percentage error to the nearest integer by which the exact result for T differs from the simple formula $T_0 = 2\pi \sqrt{\frac{L}{g}}$ for a 60° degree angle. What is this percentage error corresponding to the fractional error $\frac{T - T_0}{T_0}$? What is the exact value of $f(x)$ to 4 decimal places for a 60° degree angle?
- f) If you examine the list of values $T_{2n}(0.5)$, at what value of $2n$ does the Taylor approximation at this angle round off to the 4 decimal place value of the exact correction factor? Be sure to justify this conclusion by stating at least the 5 decimal place value of $T_{2n}(0.5)$ for both $2n$ and $2(n - 1)$.

► solution