MAT2705-04/05 22S Take Home Test 3 Print Name (Last, First) ____________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad). **You may use technology for row reductions, matrix inverses, determinants, plotting and root finding without showing intermediate steps.** ***Print*** the requested technology plots, labeling them and **annotating them appropriately *by hand*** and attach to the end of your test. All differential equations should be solved "by hand" unless otherwise specified.

pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have read the long instructions on the class web page. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date:

1. The displacement $x(t)$ of a damped harmonic oscillator system satisfies
$$m\,x''(t) + c\,x'(t) + K\,x(t) = F(t).$$
a) Let $m=2$, $c=8$, $K=400$, $F(t)=0$ and consider the initial conditions $x(0)=4$, $x'(0)=20$.
Give both exact and numeric values to 3 decimal places for the following quantities.
What are the natural frequency ω_0 , natural decay time τ_0 , the quality factor $Q=\omega_0\tau_0$ and the natural period $T_0=2\,\pi/\omega_0$ for this system?
b) Find the underdamped general solution step by step. What are the solution frequency ω_1 and decay constant τ_1? [The subscripts distinguish these from the driven system parameters in part g).]
c) Evaluate the initial value problem solution step by step by hand. [Before proceeding, check the Maple solution to know your goal.]
d) Express the sinusoidal factor in phase-shifted cosine form with the phase shift between $-\pi$ and π exactly and to 3 decimal places. What fraction of a cycle does this represent? What angle in degrees, to the nearest degree?
e) Evaluate the envelope functions for this decaying oscillation and plot your solution together with the envelope functions for an ***appropriate*** decay window starting at $t=0$. [**plot#1**]
f) Let $m=2$, $c=8$, $K=400$, $F(t)=2\sin(15\,t)$ and evaluate the steady state solution by hand and its amplitude A_{15} exactly and numerically to 4 decimal places. First write the conditions determining the undetermined coefficients in matrix form and solve using the inverse matrix.
g) Let $m=2$, $c=8$, $K=400$, $F(t)=2\sin(\omega\,t)$, $\omega \geq 0$ and again evaluate the steady state solution by hand to explore resonance. Use calculus to find the peak frewquency ω_{peak}. Evaluate also the ratio $A(\omega_{peak})/A(0)$ and compare with Q.
h) Does your value of $A(15)$ agree with part f) as it should? [Find your error if not!]

2. $x_1'(t) = -4\,x_1(t) + 2\,x_2(t)$, $x_2'(t) = 3\,x_1(t) - 5\,x_2(t)$, $x_1(0) = -7$, $x_2(0) = 3$.
a) Identify the coefficient matrix for this system of DEs and use Maple's Eigenvector result to write down a new basis for R^2 consisting of eigenvectors $\vec{b}_1$, $\vec{b}_2$ of this matrix with integer components. Order the real eigenvalues $|\lambda_1| \leq |\lambda_2|$ by increasing absolute value. Identify $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$ and B^{-1}.
b) Evaluate the new coordinates $\langle y_1, y_2 \rangle$ of the point $\langle x_1, x_2 \rangle = \langle -7, 3 \rangle$ with respect to this basis of eigenvectors

using matrix mulitplication.
c) Using the eigenbasis, convert the system to the new variables and solve this initial value problem by hand, showing all steps, and giving the final scalar form of the expressions for the old variables. Make sure the final result agrees with Maple.
d) Use the Maple template worksheet to plot a directionfield for this DE with the solution curve (for $t \geq 0$) through the initial data point **[plot#2]**, and (by hand if necessary) include the lines through the two eigenvectors representing the two subspaces of eigenvectors. Choose an appropriate window that shows everything clearly without wasting additional window space, except to see the new coordinate axes extend slightly beyond the details of the plot. [The template has an appropriate window already chosen.] By hand label these lines by their new coordinate labels, draw in and label the eigenvectors and the initial data vector $\vec{\boldsymbol{x}}(0)$ themselves as arrows, and include the parallelogram projection of the latter vector onto the new coordinate axes, i.e., draw the parallelogram parallel to the new coordinate axes with the initial data vector as its main diagonal.
e) Which of the two characteristic times in this problem is longer and hence determines the timescale on which the asymptotic limit is "reached".

3. Consider the following DE system IVP:
$x_1' = -5\,x_1 - 5\,x_2 - 2\,x_3,\ x_2' = 6\,x_1 + 6\,x_2 + 5\,x_3,\ x_3' = -6\,x_1 - 6\,x_2 - 5\,x_3,\quad x_1(0) = 1,\ x_2(0) = 0,\ x_3(0) = 1$.
a) Write down the DE system and its initial condition in explicit matrix form, identifying the coefficient matrix A.
b) Evaluate the characteristic equation and determine the ordered eigenvalues $\lambda_1(real),\ \lambda_2 = \lambda_+ ,\ \lambda_3 = \lambda_-$ (positive imaginary eigenvalue first).
c) Solve the equations using the reduction algorithm necessary to find an eigenbasis matrix $B = \langle \vec{\boldsymbol{b}}_1 | \vec{\boldsymbol{b}}_2 | \vec{\boldsymbol{b}}_3 \rangle$.
d) Re-express the DE system in the new variables $\vec{\boldsymbol{y}} = B^{-1}\vec{\boldsymbol{x}}$ and solve the resulting decoupled equations, then write out the vector form of the general solution for the original vector variable.
e) Re-express the complex eigenvalue contributions to your solution as an explicitly real pair of modes.
f) Impose the initial conditions on your general solution, using a matrix inverse to accomplish this and express the final solution in scalar form $x_1 =$..., *etc.*
g) Plot the three solution curves with the three equilibrium lines versus t for $t = 0\,..T$ for an appropriate decay window which shows clearly their asymptotic behavior, adjusting your time interval to see the merging to the equilibrium solution.
Label the three graphs by their variable names **[plot#3]**. Justify your initial choice of time interval.

Advice. When in doubt about how much work to show, show more. Explain using words if it helps. Think of this take-home test as an exercise in "writing intensive" technical expression. Try to impress bob as though it were material for a job interview. In a real world technical job, you need to be able to write coherent technical reports that other people can follow. Print out your three plots and hand annotate them, labeling axes and key points by hand. Example plot:

$$> \ plot\left(\left[2\cos(5\,t) + \sin(5\,t), \sqrt{5}, -\sqrt{5}\right], t = 0\,..2\cdot\frac{2\,\pi}{5}, color = [red, blue, blue]\right)$$