

MAT270504/05 225 Final Exam Answers (1)

$$\text{C) } \begin{aligned} x_1'' &= -10x_1 + 2x_2 + 120 \cos t & x_1(0) &= 15, \quad x_1'(0) = 0 \\ x_2'' &= 3x_1 - 15x_2 & x_2(0) &= 7, \quad x_2'(0) = 0 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -10 & 2 \\ 3 & -15 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 120 \cos t \\ 0 \end{bmatrix}}_f, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$q) \quad x_1''(t) = -10x_1(t) + 2x_2(t) + 120 \cos(t), \\ x_2''(t) = 3x_1(t) - 15x_2(t), \quad x_1(0) = 15, \quad x_2(0) = 7, \quad x_1'(0) = 0, \quad x_2'(0) = 0$$

$$\begin{aligned} \hookrightarrow x_1(t) &= 14 \cos(\omega_1 t) + 2 \cos(3t) - \cos(4t), \\ x_2(t) &= 3 \cos(\omega_2 t) + \cos(\beta t) + 3 \cos(4t) \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{using context sensitive menu}$$

b) $\boxed{\begin{array}{lll} \omega_3 = 1 & \omega_1 = 3 & \omega_2 = 4 \\ T_3 = 2\pi/1 & T_1 = 2\pi/3 & T_2 = 2\pi/4 \\ \approx 6.283 & \approx 2.094 & \approx 1.571 \end{array}}$

d) Eigenvectors (A) : $\begin{bmatrix} -16 \\ -9 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} & 2 \\ 1 & 1 \end{bmatrix}$

$$e) \quad 0 = |A - \lambda I| = \begin{vmatrix} -10-\lambda & 2 \\ 3 & -15-\lambda \end{vmatrix} = (\lambda+10)(\lambda+15) - 6 = \lambda^2 + 25\lambda + 150 - 6 = \lambda^2 + 25\lambda + 144$$

$(\lambda_1 < \lambda_2)$

$$\lambda = \frac{-25 \pm \sqrt{25^2 - 4(144)}}{2} = \frac{-25 \pm \sqrt{625 - 576}}{2} = \frac{-25 \pm \sqrt{49}}{2} = \frac{-25 \pm 7}{2} = \begin{cases} -9 \\ -16 \end{cases} = -\omega^2$$

$$\lambda = -9: A + 9I = \begin{bmatrix} -10+9 & 2 \\ 3 & -15+9 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = 2x_2 = 2t$$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\lambda = -16: A + 16I = \begin{bmatrix} -10+16 & 2 \\ 3 & -15+16 \end{bmatrix} = \begin{bmatrix} +6 & 2 \\ 3 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow x_1 = -\frac{1}{3}x_2 = -t, x_2 = t$$

$$B = \langle b_1, b_2 \rangle = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{6+1} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad A_B = B^{-1}AB = \begin{bmatrix} -9 & 0 \\ 0 & 16 \end{bmatrix}$$

$$f) \quad \begin{aligned} <2,1> \rightarrow M_1 &= 1/2 \\ <-1,3> \rightarrow M_2 &= 3/(-1) = -3 \end{aligned} \quad \begin{aligned} y &= \frac{1}{2}x \quad (\lambda = -9 \text{ eigenline}) \\ y &= -3x \quad (\lambda = -16 \text{ eigenline}) \end{aligned}$$

MAT2705-04/05 22S Final Exam Answers (2)

$$g) \quad x'' = Ax + f \xrightarrow{x=By} (By)'' = A(By) + f \rightarrow y'' = B^{-1}(By)'' = \underbrace{B^{-1}A}_{AB} By + B^{-1}f$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -9 & 0 \\ 0 & -16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 360/7 \cos t \\ -120/7 \cos t \end{bmatrix}$$

$$B^{-1}f = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 120 \cos t \\ 0 \end{bmatrix} = \frac{120}{7} \cos t \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{cases} y_1'' = -9y_1 + \frac{360}{7} \cos t \\ y_2'' = -16y_2 - \frac{120}{7} \cos t \end{cases} \quad \text{decoupled equations} \rightarrow \begin{aligned} y_1'' + 9y_1 &= \frac{360}{7} \cos t \\ y_2'' + 16y_2 &= -\frac{120}{7} \cos t \end{aligned}$$

$$y_{1h} = C_1 \cos 3t + C_2 \sin 3t$$

$$y_{2h} = C_3 \cos 4t + C_4 \sin 4t$$

$$g) [y_{1p} = C_5 \cos t]$$

$$1) [y_{4p}'' = -C_5 \cos t]$$

$$y_{1p}'' + 9y_{4p} = (9-1)C_5 \cos t = \frac{8C_5}{7} \cos t$$

$$C_5 = \frac{1}{8} \frac{360}{7} = \frac{45}{7}$$

$$y_{1p} = \frac{45}{7} \cos t$$

$$16[y_{2p} = C_6 \cos t]$$

$$2) [y_{2p}'' = -C_6 \cos t]$$

$$y_{2p}'' + 16y_{2p} = (16-1)C_6 \cos t = -\frac{15C_6}{7} \cos t$$

$$C_6 = -\frac{120}{15 \cdot 7} = -\frac{40}{35} = -\frac{8}{7}$$

$$y_{2p} = -\frac{8}{7} \cos t$$

$$\begin{cases} y_1 = C_1 \cos 3t + C_2 \sin 3t + \frac{45}{7} \cos t \\ y_2 = C_3 \cos 4t + C_4 \sin 4t - \frac{8}{7} \cos t \end{cases}$$

general soln
decoupled
equations.

y_{1h}

y_{1p}

$$h) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C_1 \cos 3t + C_2 \sin 3t + \frac{45}{7} \cos t \\ C_3 \cos 4t + C_4 \sin 4t - \frac{8}{7} \cos t \end{bmatrix}$$

general
solution

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -3C_1 \sin 3t + 3C_2 \cos 3t - \frac{45}{7} \sin t \\ -4C_3 \sin 4t + 4C_4 \cos 4t + \frac{8}{7} \sin t \end{bmatrix}$$

$$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C_1 + \frac{45}{7} \\ C_3 - \frac{8}{7} \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \end{bmatrix} \rightarrow \begin{bmatrix} C_1 + \frac{45}{7} \\ C_3 - \frac{8}{7} \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ 7 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 45+7 \\ -15+14 \end{bmatrix} = \begin{bmatrix} 52/7 \\ -1/7 \end{bmatrix} \rightarrow \begin{cases} C_1 = \frac{52-45}{7} = \frac{7}{7} = 1 \\ C_3 = -\frac{1}{7} + \frac{8}{7} = \frac{7}{7} = 1 \end{cases}$$

$$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3C_2 \\ 4C_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3C_2 \\ 4C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} C_2 = 0 \\ C_4 = 0 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \cos 3t + \frac{45}{7} \cos t \\ \cos 4t - \frac{8}{7} \cos t \end{bmatrix} = \begin{bmatrix} 2 \cos 3t - \cos 4t + \left(\frac{45}{7}(2) + \frac{8}{7}\right) \cos t \\ \cos 3t + 3 \cos 4t + \left(\frac{45}{7} - \frac{8}{7}(3)\right) \cos t \end{bmatrix}$$

$$x_1 = 2 \cos 3t - \cos 4t + 14 \cos t, \quad x_2 = \cos 3t + 3 \cos 4t + 3 \cos t$$

success!

MATZ05-0405 22S Final Exam Answers (3)

i) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\cos 3t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \cos 4t \begin{pmatrix} -1 \\ 3 \end{pmatrix}}_{X_h} + \underbrace{\cos t \begin{pmatrix} 14 \\ 3 \end{pmatrix}}_{X_p} \quad \downarrow$

$a_3 = \begin{pmatrix} 14 \\ 3 \end{pmatrix} \leftarrow \begin{matrix} \text{same signs} \\ \text{so tandem mode} \end{matrix}$

j) $x_h(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$X_p(0)$

$$= b_1 + b_2$$

g) Find by hand the general solution of the corresponding decoupled system of DEs $\vec{y}'' = A_B \vec{y} + B^{-1} \vec{F}$. First write these equations out in explicit matrix form, then obtain the two equivalent scalar DEs which are its components. Then solve them to find their general solutions using the method of undetermined coefficients. State your general solutions in scalar form and box them: $y_1(t) = \dots$, $y_2(t) = \dots$, identifying the homogeneous and particular parts of each solution: $y_1 = y_{1h} + y_{1p}$, $y_2 = y_{2h} + y_{2p}$.

h) Then express the general solution for $\vec{x} = B \vec{y}$ in explicit matrix form (without multiplying matrix factors) **and impose the initial conditions** using matrix methods (inverse matrix) to solve the linear systems. Write out and box the final scalar solutions: $x_1(t) = \dots$, $x_2(t) = \dots$. Do they agree with Maple's solution from part a)? If not, look for your error. Did you input the equations correctly?

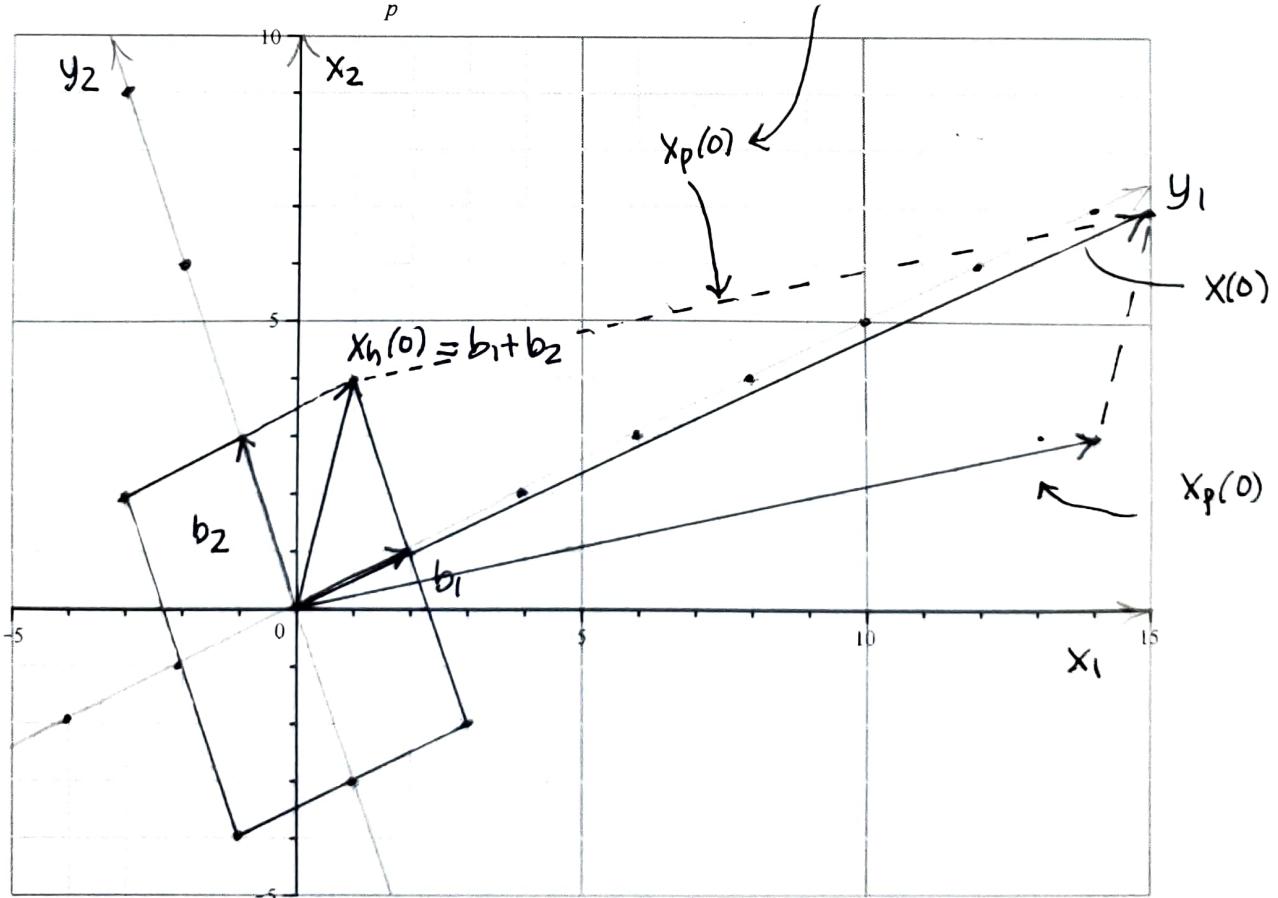
i) Express the (correct) solution as a sum of the two eigenvector modes and the response mode in the form:

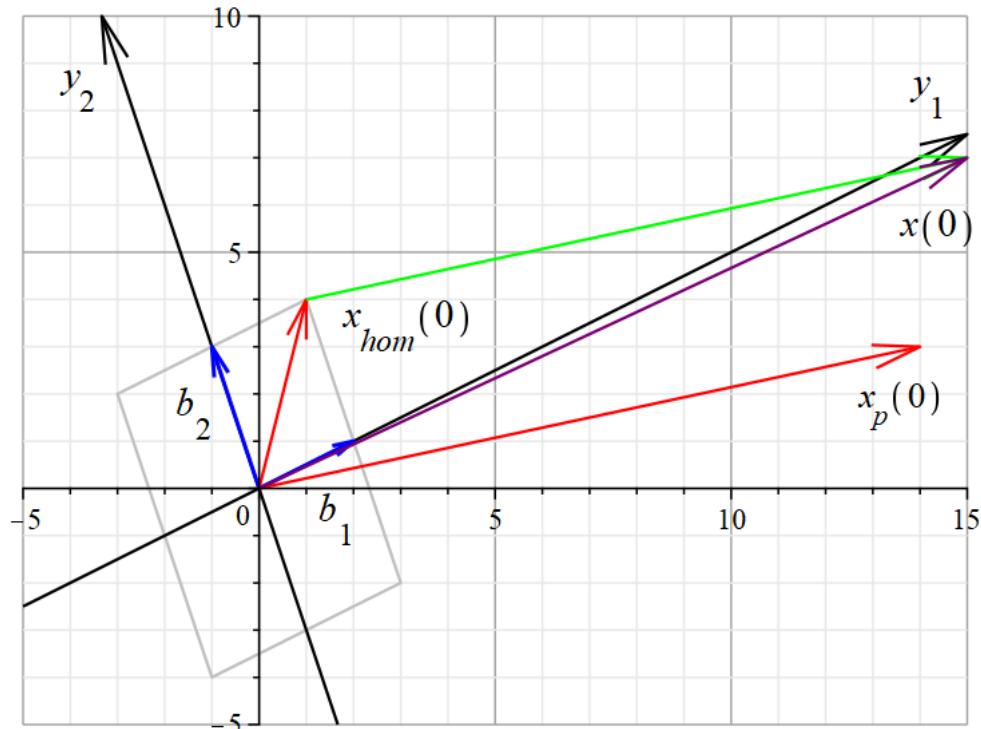
$\vec{x} = \vec{x}_{hom} + \vec{x}_p = y_{1h} \vec{b}_1 + y_{2h} \vec{b}_2 + \cos(t) \vec{a}_3$ thus identifying the particular solution \vec{x}_p (last term), the response vector coefficient \vec{a}_3 and the homogeneous solution \vec{x}_h (first two terms), as well as the final expressions for the two decoupled variables y_{1h} and y_{2h} . Is the response term a tandem or accordian mode? Explain.

j) On the grid provided, draw in the vector $\vec{x}_{hom}(0)$ and label this vector. On your graph, draw in exactly the parallelogram parallel to the new coordinate axes which project this vector along those axes and lightly shade it in in pencil (pen?). Extend the boundary of this parallelogram to the other 4 quadrants to obtain the bounding box parallelogram containing the homogeneous solution curve: its main diagonal is formed by the back to back vectors $\vec{x}_{hom}(0), -\vec{x}_{hom}(0)$.

k) Finally draw in the vectors $\vec{x}(0)$, $\vec{x}_p(0)$ and label them.

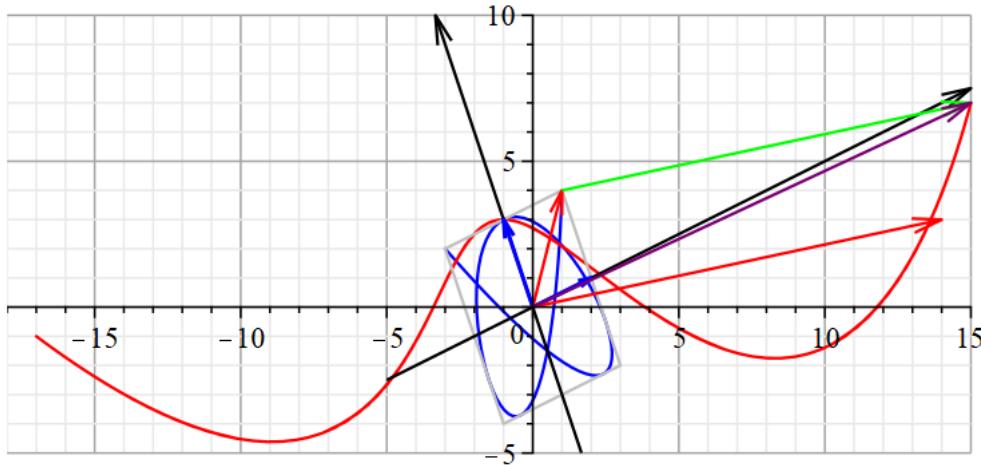
$$\text{not requested: } \vec{x}_h(v) + \vec{x}_p(v) = \vec{x}(v)$$





The green vector is $x_p(0)$ tip to tail with $x_{hom}(0)$ illustrating the vector sum $x(0) = x_{hom}(0) + x_p(0)$.

Bonus plot:



The blue curve is the homogeneous solution. The red curve is the total solution.