

MAT2705-04/05 23 S Quiz 7 bonus (do hand calculations)

Continue with the quiz example to illustrate the other homework exercises.

$$m x'' + c x' + k x = F \rightarrow x'' + k_0 x' + \omega_0^2 x = F/m$$

quiz:  $m=4=c$ ,  $k=17$ ,  $F=0$ , initial conditions  $x(0)=4$ ,  $x'(0)=-5$   
 [Note  $\omega_0 = \sqrt{17}/2 \approx 2.06$ ,  $k_0=1$ ,  $Q_0 \approx 2.06$ ]

Now:  $4x'' + 4x' + 17x = F$  nonhomogeneous case

a) Consider a forcing function  $F(t) = \cos 2t$  and impose initial conditions  $x(0)=0$ ,  $x'(0)=0$ .

First find the particular solution (steady state), then impose the initial conditions on the total solution (adding the general solution of the homogeneous case found in the quiz).

$$x_p = C_3 \cos 2t + C_4 \sin 2t \xrightarrow{\text{backsub}} \text{Find } x_p = \frac{1}{65} (\cos 2t + 8 \sin 2t)$$

$$\text{Note } A_p = \frac{\sqrt{1+64}}{65} = \frac{1}{\sqrt{65}}, \underbrace{\delta_p = \arctan\left(\frac{8}{1}\right)}_{\text{nearly } 90^\circ} \text{ so } x_p = \frac{1}{\sqrt{65}} \cos(2t - \arctan 8) \\ (\text{because } 2 \text{ is close to the natural frequency})$$

$$\text{Then } x = e^{-t/2} (C_1 \cos 2t + C_2 \sin 2t) + \frac{1}{65} (\cos 2t + 8 \sin 2t)$$

$$\hookrightarrow \text{impose inits to find } (C_1, C_2) = \frac{1}{65} (-1, -\frac{33}{4})$$

$$\text{so } A_0 = \frac{1}{65} \sqrt{1 + \left(\frac{33}{4}\right)^2} = \frac{\sqrt{1105}}{260}, \quad \delta_0 = -\pi + \arctan\left(\frac{33}{4}\right)$$

$$x_h = \frac{\sqrt{1105}}{260} \cos(2t - \pi + \arctan \frac{33}{4}) e^{-t/2} \text{ (transient.)}$$

b) Now let  $F = \cos 2\omega t$  (so  $B_0 = 1/4$  in our general formulas)  
 And find the frequency at which the response amplitude is maximum (resonance) and the amplitude there, so both  $\omega_{peak}$  and  $A(\omega_{peak})$  using the formula  $A(\omega) = \frac{B_0}{\sqrt{(\omega^2 - \omega_0^2)^2 + k_0^2 \omega^2}}$ .

$$\text{Note } A(2) = \frac{1}{\sqrt{65}} \text{ reproducing previous result. } (A(2) \approx 0.124) \leftarrow$$

$$\text{you will find } (\omega_{peak}) = \frac{\sqrt{17}}{2} \approx 1.936, \quad A(\omega_{peak}) = \frac{1}{8} = 0.125 \leftarrow$$

c) Now set damping coefficient to zero ( $k_0 \rightarrow 0$ ) and resolve part a), namely:

$$4x'' + 17x = \cos 2t, \quad x(0)=0, \quad x'(0)=0$$

$$\text{you will find } x = \cos 2t - \cos\left(\frac{\sqrt{17}}{2}t\right) = \frac{2 \sin\left(\frac{\sqrt{17}}{4}-1\right)t \sin\left(\frac{\sqrt{17}}{4}+1\right)t}{\pm A(t) \omega} \quad \omega = \frac{\sqrt{17}}{2}$$

then plot  $x$  with  $\pm A(t)$  for  $t = 0.. \frac{2\pi}{\omega}$

to see two beats (dramatic since  $\frac{\sqrt{17}}{2} \approx 2.06$  close to 2.0)